Local and global proper infiniteness for continuous C(X)-algebras

Etienne Blanchard

LOCAL AND GLOBAL PROPER INFINITENESS FOR CONTINUOUS $C(X)$-ALGEBRAS

ETIENNE BLANCHARD

Abstract. All unital continuous C*-bundles with properly infinite fibres are properly infinite C*-algebras if and only if the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ of two copies of the Cuntz extensions $\mathcal{T}_2$ generated by two isometries with orthogonal ranges is a K_1-injective C*-algebra ([BRR08, Theorem 5.5], [Blan10, Proposition 4.2]). We show that for all integer $n \geq 3$, there is a state $\psi_n : \mathcal{T}_n \to \mathbb{C}$ such that the reduced unital free product $(\mathcal{T}_n, \psi_n) *_{\mathbb{C}} (\mathcal{T}_n, \psi_n)$ is a K_1-injective C*-algebra which contains the algebraic free product $\mathcal{T}_n \circledast_{\mathbb{C}} \mathcal{T}_n$.

1. Introduction

The classification programme of nuclear C*-algebras through K-theoretical invariants launched by G. Elliott ([Ell94]) led A. Toms and W. Winter to introduce the strong self-absorption property for simple unital C*-algebras ([TW07]). This notion is pretty rigid: Any separable unital continuous $C(X)$-algebra A the fibres of which are isomorphic to the same strongly self-absorbing C*-algebra D is a trivial $C(X)$-algebra provided the compact Hausdorff base space X has finite topological dimension. (Indeed, the strongly self-absorbing C*-algebra D tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$ ([Win09]). Hence, this C*-algebra D is K_1-injective ([Ror04]) and the $C(X)$-algebra A satisfies $A \cong D \otimes C(X)$ ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C*-bundle over the infinite dimensional compact product $\Pi_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2^∞ ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dad09, §3] for all pair (Γ_0, Γ_1) of discrete countable torsion groups a unital separable continuous $C(X)$-algebra A such that:

– the base space X is the compact Hilbert cube $X = \mathfrak{X}$ of infinite dimension,
– all the fibres A_x $(x \in \mathfrak{X})$ are isomorphic to the strongly self-absorbing Cuntz C*-algebra $\mathcal{O}_2$ generated by two isometries s_1, s_2 satisfying $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$,
– $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$, where $Y_0 \subset [0,1]$ is the canonical Cantor set.

These K-theoretical conditions imply that the $C(\mathfrak{X})$-algebra A is not a trivial one. But this argument does not anymore work when the strongly self-absorbing algebra D is the Cuntz algebra $\mathcal{O}_\infty$ ([Cun77]), in so far as $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ is a torsion free group.

In this note, we tackle this trivialization problem for unital continuous $C(X)$-algebras with fibres $\mathcal{O}_\infty$ in a different way: All unital continuous $C(X)$-algebras with properly infinite fibres are properly infinite C*-algebras if and only if the full unital free product

2010 *Mathematics Subject Classification.* Primary: 46L80; Secondary: 46L06, 46L35.
Key words and phrases. C*-algebra, Classification, Proper Infiniteness.

$\mathbf{A} := \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ of two distinct copies of the unital extension $\mathcal{T}_2$ of the Cuntz algebras $\mathcal{O}_2$ by the compact operators ([Cun77]) is a K_1-injective C*-algebra, *i.e.* the canonical map from $\mathcal{U}(\mathbf{A})/\mathcal{U}^0(\mathbf{A})$ to $K_1(\mathbf{A})$ is injective (Corollary 4.2). That C*-algebra $\mathbf{A}$ unitally embeds in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_3$ ([ADEL04]) and works by M. Rieffel ([Rief83], [Rief87]), K. Dykema, U. Haagerup, M. Rørdam ([DHR97], [Roh09]) enable us to prove that some reduced quotient $\mathfrak{B}$ of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_3$ is a unital K_1-injective C*-algebra which already contains the algebraic unital free product $\mathcal{T}_2 \circledast_{\mathbb{C}} \mathcal{T}_3$ of the two Cuntz extensions $\mathcal{T}_2$ and $\mathcal{T}_3$ (Proposition 5.2).

I especially thank E. Kirchberg for a few inspiring remarks.

2. Notations

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of a subset S in a Banach space.

Definition 2.1. *(*[Dix69], [Kas88], [Blan97]*) Let X be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous function on X.*

- *A unital $C(X)$-algebra is a unital C*-algebra A endowed with a unital morphism of C*-algebra from $C(X)$ to the centre of A.*
- *For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a_{|F}$ the image of a in the quotient $A_{|F} := A/C_0(X \setminus F) \cdot A$. If $x \in X$ is a point in X, one calls fibre at x the quotient $A_x := A_{|\{x\}}$ and one write a_x for $a_{|\{x\}}$.*
- *The $C(X)$-algebra A is said to be continuous if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.*

Definition 2.2. *(*[Pim95]*) Let X be a compact Hausdorff space.*
a) The full Fock Hilbert $C(X)$-module $\mathcal{F}(E)$ of a Hilbert $C(X)$-module E is the direct sum of Hilbert $C(X)$-module

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(\mathfrak{X})})\, m}, \tag{2.1}$$

where $E^{(\otimes_{C(\mathfrak{X})})\, m} := \begin{cases} C(X) & \text{if } m = 0\,, \\ E \otimes_{C(X)} \ldots \otimes_{C(X)} E \ \ (m \text{ terms}) & \text{if } m \geq 1\,. \end{cases}$

b) The Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ of a full Hilbert $C(X)$-module E, i.e. with non-zero fibres, is the unital subalgebra of the $C(X)$-algebra $\mathscr{L}_{C(\mathfrak{X})}(\mathcal{F}(E))$ of adjointable $C(X)$-linear operator acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(\zeta)$ ($\zeta \in E$), where:

$$\begin{array}{llll} - & \ell(\zeta)\,(f \cdot \hat{1}_{C(\mathfrak{X})}) & := f \cdot \zeta = \zeta \cdot f \quad \textit{for} \quad f \in C(X) & \text{and} \\ - & \ell(\zeta)\,(\zeta_1 \otimes \ldots \otimes \zeta_k) & := \zeta \otimes \zeta_1 \otimes \ldots \otimes \zeta_k \quad \textit{for} \quad \zeta_1, \ldots, \zeta_k \in E & \text{if } k \geq 1\,. \end{array} \tag{2.2}$$

Remarks 2.3. a) ([Cun81], [BRR08]) For all integer $n \geq 2$, the C*-algebra $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$ is the universal unital C*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1 s_1^* + \ldots + s_n s_n^* \leq 1\,. \tag{2.3}$$

b) A unital C*-algebra A is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Ror03, Proposition 2.1]):

– A contains two isometries with mutually orthogonal range projections,
– A contains a unital copy of the simple Cuntz C*-algebra $\mathcal{O}_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges.

3. Local proper infiniteness

Let $(C^*(\mathbb{Z}),\Delta)$ be the unital Hopf C*-algebra ([Wor95]) generated by a unitary $\mathbf{u}$ with spectrum $\mathbb{T}:=\{z\in\mathbb{C}\,;\,z^*z=1\}$ (often written S^1) and with coproduct $\Delta(\mathbf{u})=\mathbf{u}\otimes\mathbf{u}$.

If X is a compact Hausdorff space and E is a separable Hilbert $C(X)$-module with non-zero fibres, there is only one coaction α_E of $(C^*(\mathbb{Z}),\Delta)$ on the Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ such that $\alpha_E\big(\ell(\zeta)\big)=\ell(\zeta)\otimes\mathbf{u}$ for all $\zeta\in E$, *i.e.*

$$\alpha_E:\begin{array}{ccccc}\mathcal{T}(E)&\to&\mathcal{T}(E)\otimes C^*(\mathbb{Z})&=&C(\mathbb{T},\mathcal{T}(E))\\ \ell(\zeta)&\mapsto&\ell(\zeta)\otimes\mathbf{u}&=&(z\mapsto\ell(z\zeta))\end{array}\tag{3.1}$$

The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^{\alpha_E}=\{a\in\mathcal{T}(E)\,;\,\alpha_E(a)=a\otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^{\alpha_E}=\big[C(X).1+\sum_{k\geq 1}\ell(E)^k\cdot\big(\ell(E)^k\big)^*\big]\,.\tag{3.2}$$

Besides, the following local proper infiniteness property holds.

Proposition 3.1. *(*[Blac04]*) Let X be a compact Hausdorff space and let E be a separable Hilbert $C(X)$-module the fibres of which all have dimension greater than 2.*

If x is a point in X, then there exists a closed neighbourhood $F\subset X$ of x such that the restriction $\mathcal{T}(E)_{|F}$ of the Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ is properly infinite.

Proof. Let ζ_1 and ζ_2 be two norm 1 sections in E satisfying $\big[\langle\zeta_i,\zeta_j\rangle(x)\big]=1_{M_2(\mathbb{C})}$, *i.e.* $\{(\zeta_1)_x,(\zeta_2)_x\}$ is an orthonormal system in the Hilbert space E_x. By continuity, the matrix $\big[\langle\zeta_i,\zeta_j\rangle(y)\big]\in M_2(\mathbb{C})$ is invertible for all y in a neighbourhood of x and one can even suppose that $\big[\langle\zeta_i,\zeta_j\rangle(y)\big]=1_{M_2(\mathbb{C})}$ for all points y in a closed neighbourhood $F\subset X$ of the point x, up to replacing the section $\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$ by $c\cdot\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$ for some continuous function $c\in C(X\,;\,M_2(\mathbb{C}))$ satisfying $c(x)=1_{M_2(\mathbb{C})}$. This orthonormalization procedure means that

$$\ell(\zeta_i)^*\ell(\zeta_j)_{|F}=\langle\zeta_i,\zeta_j\rangle_{|F}=\delta_{i=j}\cdot 1_{|F}\qquad\text{for all } i,j=1\text{ or }2\,.$$

Thus, the restricted unit $1_{|F}=\ell(\zeta_1)^*\ell(\zeta_1)_{|F}=\ell(\zeta_2)^*\ell(\zeta_2)_{|F}$ is greater that the sum $\ell(\zeta_1)\ell(\zeta_1)^*{}_{|F}+\ell(\zeta_2)\ell(\zeta_2)^*{}_{|F}$ and is therefore a properly infinite projection in the Pimsner-Toeplitz algebra $\mathcal{T}(E)_{|F}$. ☐

Remarks 3.2. a) One can also prove Proposition 3.1 thanks to the semiprojectivity of the Cuntz C*-algebra $\mathcal{O}_\infty$ ([Blac04, Theorem 3.2]).

b) The restriction $\mathcal{T}(E)_{|F}$ is properly infinite if the closed subset $F\subset X$ is *perfect* (*i.e.* without any isolated point) and there is a section $\zeta\in E$ with $\|\zeta_x\|\geq 1$ for every point $x\in F$.

4. Global proper infiniteness

Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following global version of Proposition 3.1 on proper infiniteness for continuous $C(X)$-algebras with properly infinite fibres.

Proposition 4.1. *Let X be a second countable perfect compact Hausdorff space and let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres.*

1) There exist:

- (a) *a finite integer $n \geq 1$,*
- (b) *a covering $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_n}$ by the interiors of closed balls $F_1, \ldots, F_n$,*
- (c) *unital embeddings of C*-algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A_{|F_k}$ $(1 \leq k \leq n$).*

2) The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integers p.

3) For all integers i, j in $\{1, \ldots n\}$, there is a unitary $u_{i,j} \in \mathcal{U}(A_{|F_i \cap F_j})$ such that

$$\sigma_i(s_m)_{|F_i \cap F_j} = u_{i,j} \cdot \sigma_j(s_m)_{|F_i \cap F_j} \quad \text{for all } m \in \mathbb{N}.$$

Proof. 1) For all point $x \in X$, the semiprojectivity of the C*-subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([Blac04, Theorem 3.2]) implies that there are a closed neighbourhood $F \subset X$ of the point x and a $C(F)$-linear unital embedding $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A_{|F}$. The compactness of the topological space X enables to conclude.

2) derives from Proposition 3.1, [BRR08, Proposition 2.7] and [Ror97, Proposition 2.1].

3) Set $u_{i,j} := \sum_{m \in \mathbb{N}} \sigma_i(s_m)_{|F_i \cap F_j} \cdot \sigma_j(s_m)^*{}_{|F_i \cap F_j}$ ([Blan13, Proposition 6.3]). □

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the C*-algebra A is properly infinite. Indeed, there exists a unital C*-algebra A which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is properly infinite ([Ror03, Proposition 4.5]). We nevertheless have the following corollary.

Corollary 4.2. *Let $\jmath_0, \jmath_1$ denote the two canonical unital embeddings of the C*-algebra $\mathcal{T}_2$ in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ and let $u \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a K_1-trivial unitary satisfying $\jmath_1(s_1) = u \cdot \jmath_0(s_1)$ (*[BRR08, Lemma 2.4]*).*

Then the following conditions are equivalent:

- (a) *The full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective.*
- (b) *The unitary $u \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ of the unit $1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2}$.*
- (c) *Every separable unital continuous $C(X)$-algebra A with properly infinite fibres is a properly infinite C*-algebra.*

Proof. (a)⇒(b) A unital C*-algebra A is K$_1$-injective if and only if every unitary $v \in \mathcal{U}(A)$ is homotopic to the unit 1_A in $\mathcal{U}(A)$. Hence, (b) is a special case of (a).

(b)⇒(c) Let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \to A_{|F_k}$ $(1 \leq k \leq n)$. Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $1 \leq k \leq n$ and let us construct by induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

– If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

– If $k \in \{1, \dots, n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [BRR08] implies that there exist an homomorphism of unital C*-algebra $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to A_{|G_k \cap F_{k+1}}$ and a K$_1$-trivial unitary $u_{k+1} \in \mathcal{U}(A_{|G_k \cap F_{k+1}})$ satisfying:

- $\pi_k(\jmath_0(s_1)) = w_{k|G_k \cap F_{k+1}}$ and
- $\pi_k(\jmath_1(s_1)) = \sigma_{k+1}(s_1)_{|G_k \cap F_{k+1}} = u_{k+1} \cdot w_{k|G_k \cap F_{k+1}}$.

If condition (b) is true, this unitary u_{k+1} is homotopic to $1_{A_{|G_k \cap F_{k+1}}}$ in $\mathcal{U}(A_{|G_k \cap F_{k+1}})$, *i.e.* $u_{k+1} \in \mathcal{U}^0(A_{|G_k \cap F_{k+1}})$, so that it admits a unitary lifting z_{k+1} in $\mathcal{U}^0(A_{|F_{k+1}})$ (see *e.g.* [LLR00, Lemma 2.1.7]). The only isometry $w_{k+1} \in A_{|G_{k+1}}$ satisfying the two constraints:

$$w_{k+1|G_k} = w_k \quad \text{and} \quad w_{k+1|F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)$$

verifies that the two projections $w_{k+1} w_{k+1}^*$ and $1_{|G_{k+1}} - w_{k+1} w_{k+1}^*$ are properly infinite and full in $A_{|G_{k+1}}$.

The proper infiniteness of the projection $w_n w_n^* \in A_{|G_n} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$ is also a properly infinite projection in A, *i.e.* the C*-algebra A is properly infinite.

(c)⇒(a) The C*-algebra $\mathcal{D} := \{ f \in C([0,1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) \,;\, f(0) \in \jmath_0(\mathcal{T}_2) \text{ and } f(1) \in \jmath_1(\mathcal{T}_2) \}$ is a unital continuous $C([0,1])$-algebra the fibres of which are all properly infinite. Thus, condition (c) implies that the C*-algebra $\mathcal{D}$ is properly infinite, a statement which is equivalent to the K$_1$-injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([Blan10, Proposition 4.2]). □

There is an extra grading property for Pimsner-Toeplitz algebras in case the unital C*-algebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K$_1$-injective.

Corollary 4.3. *Let X be a second countable compact Hausdorff metrizable space and let E be a separable Hilbert $C(X)$-module all of whose fibres are Hilbert spaces of dimension greater than 2.*

If the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective, then there exists an isometry w in the Pimsner-Toeplitz algebra $\mathcal{T}(E)$ such that

- $\mathcal{T}(E) = C^* < \mathcal{T}(E)^{\alpha_E}, w >$,
- *the projections ww^* and $1 - ww^*$ are properly infinite and full in $\mathcal{T}(E)$,*
- $\alpha_E(w) = w \otimes \mathbf{u}$.

Proof. Let $X = \overset{o}{F_1} \cup \ldots \cup \overset{o}{F_n}$ by a finite covering of the second countable compact Hausdorff space X by interiors of closed subsets $F_1, \dots F_n$ and let $\zeta_1, \zeta_2, \dots, \zeta_{2n}$ be $2n$ norm 1 sections in E such that $\langle \zeta_{2k-i}, \zeta_{2k-j} \rangle(y) = \delta_{i=j}$ for all indices $k \in \{1, \dots, n\}$, $0 \le i, j \le 1$ and all points $y \in F_k$.

Set $G_k := F_1 \cup \ldots \cup F_k \subset X$ for all $k \in \{1, \dots, n\}$. Then one constructs by induction isometries $w_k \in \mathcal{T}(E)_{|G_k}$ $(1 \le k \le n)$ such that:

(a) the projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in $\mathcal{T}(E)_{|G_k}$,

(b) $\mathcal{T}(E)_{|G_k} = C^* < \mathcal{T}(E)^{\alpha_E}{}_{|G_k}, w_k >$,
(c) $\alpha_{E_{\mathcal{T}(E)_{|G_k}}}(w_k) = w_k \otimes \mathbf{u}$.

– If $k = 1$, the isometry $w_1 = \ell(\zeta_1)_{|F_1}$ has the three wanted properties.

- If $k \in \{1, \dots, n-1\}$ and the isometry $w_k \in \mathcal{T}(E)_{|G_k}$ is already constructed, then the step (b)⇒(c) of the previous Corollary 4.2 shows that there exist a unitary $z_{k+1} \in \mathcal{U}^0(\mathcal{T}(E)_{|F_{k+1}})$ and an isometry $w_{k+1} \in \mathcal{T}(E)_{|G_{k+1}}$ satisfying

$$w_{k+1|G_k} = w_k \quad \text{and} \quad w_{k+1|F_{k+1}} = (z_{k+1})^* \cdot \ell(\zeta_{2k-1})_{|F_{k+1}} ,$$

provided the C*-algebra $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K$_1$-injective.

This isometry w_{k+1} then has the two first desired properties by the previous corollary. Besides, $\alpha_{E_{|G_{k+1}}}(w_{k+1}) = w_{k+1} \otimes \mathbf{u}$ since $\alpha_{E_{|G_k}}(w_k) = w_k \otimes \mathbf{u}$, $\alpha_{E_{|F_{k+1}}}(\ell(\zeta_{2k-1})_{|F_{k+1}}) = \ell(\zeta_{2k-1})_{|F_{k+1}} \otimes \mathbf{u}$ and $\alpha_{E_{|F_{k+1}}}(z_{k+1}) = z_{k+1} \otimes 1$. □

Remarks 4.4. a) The isometry $w \in \mathcal{T}(E)$ built in Corollary 4.3 cannot always belong to the subspace $\ell(E) \subset \mathcal{T}(E)$. For instance, if $\mathfrak{X} = [0,1]^\infty$ is the compact Hilbert cube of infinite topological dimension and $\mathcal{E}$ is the non-trivial Hilbert $C(\mathfrak{X})$-module constructed by J. Dixmier and A. Douady ([DD63], [BK04a, Proposition 3.6]), there exist by Proposition 3.1 finitely many sections $\zeta_1, \dots. \zeta_n$ in $\mathcal{E}$ such that

$$\mathcal{T}(\mathcal{E}) = C^* < \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, \ell(\zeta_1), \dots, \ell(\zeta_n) > \ . \tag{4.1}$$

But there is no $\zeta \in \sum_{1 \le k \le n} C(X) \cdot \zeta_k \subset \mathcal{E}$ such that $\mathcal{T}(\mathcal{E}) = C^* < \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, \ell(\zeta) >$ since:

– any section $\zeta \in \mathcal{E}$ satisfies $\zeta_x = 0$ for at least one point $x \in \mathfrak{X}$ ([DD63, Lemma 14], [BK04a, Proposition 3.6]) and
– $C^* < (\mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}})_x, \ell(\zeta)_x > \cong \mathcal{T}(\mathcal{E}_x)^{\alpha_{\mathcal{E}_x}} \subsetneq \mathcal{T}(\mathcal{E}_x) \cong \mathcal{O}_\infty$.

Note: If the C*-algebra $\mathcal{T}_{2\mathbb{C}} *_{\mathbb{C}} \mathcal{T}_2$ is K$_1$-injective, then there exists a n-uple $(b_1, \dots, b_n)$ in $(\mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}})^n \setminus C(X)^n$ such that $w = \ell(\zeta_1) \cdot b_1 + \ldots + \ell(\zeta_n) \cdot b_n$. Thus, there exists another n-uple $(d_1, \dots, d_n) \in (\mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}})^n$ such that $1_{C(X)} = w^* \cdot w = \sum_{k=1}^{n} d_k \cdot b_k$.

b) If $\mathfrak{A}$ denotes the universal UHF-algebra with $K_1(\mathfrak{A}) = \mathbb{Q}$, there is a non-trivial unital continuous $C(\mathfrak{X})$-algebra D with $D_x \cong \mathcal{O}_\infty \otimes \mathfrak{A}$ for all $x \in \mathfrak{X}$ ([Dad09]). But is there a non-trivial unital continuous $C(\mathfrak{X})$-algebra with fibres isomorphic to $\mathcal{O}_\infty$? Note that $K_0(\mathcal{O}_\infty \otimes \mathfrak{A}) = \mathbb{Q} \neq \mathbb{Z} = K_0(\mathcal{O}_\infty)$.

c) An automorphism of $C(X)$-algebra on the Pimsner-Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ is not always grading preserving, even if the base space X is reduced to a point, as noticed by E. Kirchberg ([KR00]) : If E is the Hilbert space $E = \ell^2(\mathbb{N})$, the purely infinite simple unital nuclear C*-algebra $\mathcal{T}(E) \cong \mathcal{O}_\infty$ has real rank zero. Accordingly, there exists a unitary $u \in \mathcal{U}(\mathcal{T}(E))$ such that $u \cdot \mathcal{T}(E)^{\alpha_E} \cdot u^* \not\subset \mathcal{T}(E)^{\alpha_E}$.

5. A K$_1$-injective subquotient

The universal Cuntz extension $\mathcal{T}_2$ generated by two isometries with orthogonal ranges unitally embeds in the Cuntz extension $\mathcal{T}_3$ ([Cun81, Lemma 3.1]). As a consequence, the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ unitally embeds in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_3$ ([ADEL04, Proposition 2.4]). We show in this section that some *reduced* quotient of

$\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_3$ is a simple K_1-injective C*-algebra which contains the algebraic unital free product $\mathcal{T}_2 \circledast_{\mathbb{C}} \mathcal{T}_3$.

Let $\ell^2(\mathbb{N})$ be the Hilbert space of all the sequences $\xi = (\xi_k) \in \mathbb{C}^{\mathbb{N}}$ which satisfy $\|\xi\|_2{}^2 = \sum_k |\xi_k|^2 < \infty$ and take an orthonormal basis $\{e_0, e_1, e_2, \dots\}$ in $\ell^2(\mathbb{N})$. Define for all integer $n \geq 2$:

- the n isometries $s_{n,1}, \dots, s_{n,n}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$ satisfying $s_{n,i} \cdot e_k = e_{nk+i}$ $(k \in \mathbb{N})$,
- the Cuntz extension $\mathcal{T}_n := C^* < s_{n,1}, \dots, s_{n,n} > \subset \mathbb{B}(\ell^2(\mathbb{N}))$ ([Cun77]),
- the state $\psi_n(a) = \frac{1}{n} \sum\limits_{i=1}^{n} \langle e_i, a \cdot e_i \rangle$ on the C*-algebra $\mathcal{T}_n \subset \mathbb{B}(\ell^2(\mathbb{N}))$,
- the unitary $u_n := s_{n,1}s_{n,2}^* + \dots + s_{n,n}s_{n,1}^* + p \in \mathcal{T}_n$, where p is the rank 1 projection $p = 1 - \sum_i s_{n,i}s_{n,i}^* = \theta_{e_0,e_0}$ in $\mathbb{B}(\ell^2(\mathbb{N}))$.

Lemma 5.1. *1) $\psi_n(a) = \psi_n(u_n^* a u_n)$ for any operator $a \in \mathcal{T}_n$ and $\psi_n((u_n)^j) = 0$ for any integer $j \in \{1, \dots, n-1\}$,*

2) The state ψ_n on $\mathcal{T}_n$ has a faithful GNS representation.

3) The restriction of the state ψ_n to the C-subalgebra $C^*(u_n) \subset \mathcal{T}_n$ is faithful.*

Proof. 1) The unitary u_n satisfies $(u_n)^n = 1$ and $(u_n)^j \cdot e_1 = e_{n-j+1}$ for all $1 \leq j \leq n$. Thus, there exists an isomorphism of C*-algebra $\alpha_n : C^*(u_n) \to C^*(\mathbb{Z}/n\mathbb{Z})$ and

$$\begin{aligned} \psi_n(a) &= \tfrac{1}{n} \sum_{i=1}^{n} \langle (u_n)^i \cdot e_1,\, a\, (u_n)^i \cdot e_1 \rangle \\ &= \tfrac{1}{n} \sum_{i=1}^{n} \langle (u_n)^{i+1} \cdot e_1,\, a\, (u_n)^{i+1} \cdot e_1 \rangle = \psi_n(u_n^* a u_n)\,. \end{aligned}$$

We also have $\psi_n\big((u_n)^j\big) = \langle e_1, (u_n)^j \cdot e_1 \rangle = \langle e_1, e_{n-j+1} \rangle = 0$ for all $j \in \{1, \dots, n-1\}$.

2) The C*-algebra $\mathcal{T}_n$ is contained in $\mathbb{B}(\ell^2(\mathbb{N}))$ and the Hilbert space $\ell^2(\mathbb{N})$ is the closure of $\mathcal{T}_n \cdot e_0$. Thus, if $a \in \mathcal{T}_n \setminus \{0\}$ is non-zero, there exists an element $b \in \mathcal{T}_n$ such that

$$\begin{aligned} 0 < \|ab \cdot e_0\|^2 = \|abs_{n,1}^* \cdot e_1\|^2 &= \langle e_1, s_{n,1}b^* \cdot a^*a \cdot bs_{n,1}^* e_1 \rangle \\ &= \sum_{i=1}^{n} \langle e_i, s_{n,1}b^* \cdot a^*a \cdot bs_{n,1}^* e_i \rangle \\ &= n \cdot \psi_n(s_{n,1}b^* \cdot a^*a \cdot bs_{n,1}^*)\,. \end{aligned}$$

3) The composition $\psi_n \circ \alpha_n^{-1}$ equals the only Haar state h_n on the finite dimensional Hopf C*-algebra $(C^*(\mathbb{Z}/n\mathbb{Z}), \Delta)$, which is faithful on $C^*(\mathbb{Z}/n\mathbb{Z})$. □

As a consequence, the following holds:

Proposition 5.2. *Let $(\mathfrak{B}, \psi) := (\mathcal{T}_2, \psi_2) *_{\mathbb{C}} (\mathcal{T}_3, \psi_3)$ be the reduced unital free product defined by D. Voiculescu (*[Voi85]*).*

1) The C-algebra $\mathfrak{B}$ contains the algebraic unital free product generated by $\mathcal{T}_2$ and $\mathcal{T}_3$*

2) The C-algebra $\mathfrak{B}$ is simple and has stable rank 1.*

3) The C-algebra $\mathfrak{B}$ is K_1-injective.*

Proof. 1) This embedding result is essentially contained in Proposition 4.2 of [DS01]. But we include here the main steps of that proof for completeness. Define:

- the tensor product C*-algebra $B := \mathcal{T}_2 \otimes \mathcal{T}_3$,
- the state $\phi := \psi_2 \otimes \psi_3$ on the C*-algebra B,
- the full countably generated Hilbert B-bimodule $E := L^2(B, \phi) \otimes B$,
- the full Fock Hilbert B-bimodule $\mathcal{F}(E) = B \oplus E \oplus (E \otimes E) \oplus \ldots$,
- the Pimsner-Toeplitz C*-algebra $\mathcal{T}(E)$ generated in $\mathscr{L}(\mathcal{F}(E))$ by the *creation* operators $\ell(\xi)$, $\xi \in E$ ([Pim95]).
- the conditional expectation $\mathfrak{E} : \mathcal{T}(E) \to B$ given by compression with the orthogonal projection from $\mathscr{F}_B(E)$ onto the first summand $B \subset \mathcal{F}(E)$ ([DS01]).

The C*-algebra B unitally embeds in $\mathcal{T}(E) \subset \mathscr{L}(\mathcal{F}(E))$ by

$$b \cdot \ell(\Lambda_\phi b_1 \otimes b_2) \cdot b' = \ell(\Lambda_\phi b b_1 \otimes b_2 b')\,.$$

Take a unitary v in the C*-subalgebra $C^*\!< \ell(\Lambda_\phi 1 \otimes 1) > \subset \mathcal{T}(E)$ such that $\mathfrak{E}(v^j) = 0$ for all non-zero integer $j \in \mathbb{Z}$ and let $\pi_2 : \mathcal{T}_2 \to \mathcal{T}(E)$, $\pi_3 : \mathcal{T}_3 \to \mathcal{T}(E)$ be the two unital $*$-morphisms $\pi_2(a) = v^2 \cdot (a \otimes 1_{\mathcal{T}_3}) \cdot v^{-2}$ and $\pi_3(a) = v^3 \cdot (1_{\mathcal{T}_2} \otimes a) \cdot v^{-3}$. Then, the algebraic amalgamated free product $\mathcal{T}_2 \circledast_{\mathbb{C}} \mathcal{T}_3$ studied by B. Blackadar ([Blac78]) and G. Pedersen ([Ped94]) is (isomorphic to) the unital $*$-algebra generated by $\pi_2(\mathcal{T}_2)$ and $\pi_3(\mathcal{T}_3)$ in $\mathcal{T}(E)$, whereas Voiculescu's reduced unital free product $(\mathcal{T}_2, \psi_2) *_{\mathbb{C}} (\mathcal{T}_3, \psi_3)$ ([Voi85]) is (isomorphic to) its closure (see *e.g.* [Blan09, Theorem 4.1]).

2) The previous Lemma 5.1 and [BD01, Theorem 1.3] imply that there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_{\mathbb{C}} (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \hookrightarrow (\mathfrak{B}, \psi) = (\mathcal{T}_2, \psi_2) *_{\mathbb{C}} (\mathcal{T}_3, \psi_3)$, if h_2 and h_3 denote the Haar states on the compact groups $(C^*(\mathbb{Z}/2\mathbb{Z}, \Delta_2)$ and $C^*(\mathbb{Z}/3\mathbb{Z}), \Delta_3)$. Hence, the reduced free product $\mathfrak{B}$ is simple ([Av82, Proposition 3.1+its corollary]) and this unital C*-algebra has stable rank 1 ([DHR97, Theorem 3.8]), *i.e.* the open subset of invertible elements is dense in $\mathfrak{B}$.

3) With the notations of [Rief83], the set $Lg_1(\mathfrak{B})$ of invertible elements is dense in the stable rank 1 C*-algebra $\mathfrak{B}$, so that this C*-algebra $\mathfrak{B}$ is K_1-injective by [Rief83, Proposition 1.6] and [Rief87, Theorem 2.10] . (A self-contained proof is also available in Theorem 3.2.11 of [Roh09].) □

Remark 5.3. For all strictly positive integer n, the groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are normal subgroups in the cyclic groups $\mathbb{Z}/2n\mathbb{Z}$ and $\mathbb{Z}/3n\mathbb{Z}$. Thus, there is a unital embedding $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_{\mathbb{C}} (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \subset (C^*(\mathbb{Z}/2n_1\mathbb{Z}), h_{2n_1}) *_{\mathbb{C}} (C^*(\mathbb{Z}/3n_2\mathbb{Z}), h_{3n_2})$ and the reduced unital free product $(\mathcal{T}_{2\cdot n_1}, \psi_{2\cdot n_1}) *_{\mathbb{C}} (\mathcal{T}_{3\cdot n_2}, \psi_{3\cdot n_2})$ is both simple and K_1-injective for all strictly positive integers n_1, n_2 .

Question 5.4. *Is the reduced unital free product $(\mathcal{T}_2, \psi_2) *_{\mathbb{C}} (\mathcal{T}_2, \psi_2)$ also K_1-injective?*

References

[ADEL04] S. Armstrong, K. Dykema, R. Exel, H. Li, *On embeddings of full amalgamated free product of C*-algebras*, Proc. Amer. Math. Soc. **132** (2004), 2019-2030.

[Av82] D. Avitzour, *Free products of C*-algebras*, Trans. Amer. Math. Soc. **271** (1982), 423–465.

[Blac78] B. Blackadar, *Weak expectation and nuclear C^*-algebras*, Indiana Univ. Math. J. **27** (1978), 1021–1026.

[Blac04] B. Blackadar, *Semiprojectivity in simple C^*-algebras*, Adv. Stud. Pure Math. **38** (2004), 1–17.

[Blan97] E. Blanchard, *Subtriviality of continuous fields of nuclear C*-algebras*, with an appendix by E. Kirchberg, J. Reine Angew. Math. **489** (1997), 133–149.

[Blan09] E. Blanchard, *Amalgamated free products of C*-bundles*, Proc. Edinburgh Math. Soc. **52** (2009), 23–36.

[Blan10] E. Blanchard, *K_1-injectivity for properly infinite C^*-algebras*, Clay Math. Proc. **11** (2010), 49–54.

[Blan13] E. Blanchard, *Continuous fields with fibres $\mathcal{O}_\infty$*, to appear in Math. Scand.

[BD01] E. Blanchard, K. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. **199** (2001), 1–19.

[BK04a] E. Blanchard, E. Kirchberg, *Global Glimm halving for C^*-bundles*, J. Op. Th. **52** (2004), 385–420.

[BK04b] E. Blanchard, E. Kirchberg, *Non-simple purely infinite C*-algebras: the Hausdorff case*, J. Funct. Anal. **207** (2004), 461–513.

[BRR08] E. Blanchard, R. Rohde, M. Rørdam, *Properly infinite $C(X)$-algebras and K_1-injectivity*, J. Noncommut. Geom. **2** (2008), 263–282.

[Cun77] J. Cuntz, *Simple C^*-Algebras Generated by Isometries*, Commun. Math. Phys. **57** (1977), 173–185.

[Cun81] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181-197.

[DD63] J. Dixmier, A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[Dix69] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthiers-Villars Paris (1969).

[DS01] K. Dykema, D. Shlyakhtenko, *Exactness of Pimsner-Toeplitz C^*-algebras* Proc. Edinb. Math. Soc. **44** (2001), 425–444.

[DHR97] K. Dykema, U. Haagerup, M. Rørdam, *The stable rank of some free product C^*-algebras*, Duke Math. J. **90** (1997), 95–121 [Correction in DMJ **94** (1998), 213].

[Dad09] M. Dădărlat. *Fiberwise KK-equivalence of continuous fields of C^*-algebras*, J. K-Theory **3** (2009), 205–219.

[DW08] M. Dădărlat, W. Winter, *Trivialization of $C(X)$-algebras with strongly self-absorbing fibres*, Bull. Soc. Math. France **136** (2008), 575-606.

[Ell94] G. Elliott, *The classification problem for amenable C^*-algebras*, Proc. Internat. Congress of Mathematicians (Zurich, Switzerland, 1994), Birkhauser Verlag, Basel (1995), 922–932.

[HRW07] I. Hirshberg, M. Rørdam, W. Winter, *$C_0(X)$-algebras, stability and strongly self-absorbing C^*-algebras*, Math. Ann. **339** (2007), 695–732.

[HR98] J. Hjelmborg, M. Rørdam, *On stability of C^*-algebras.* J. Funct. Anal. **155** (1998), 153–170.

[Kas88] G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201.

[KR00] E. Kirchberg, M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637–666.

[KR02] E. Kirchberg, M. Rørdam, *Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Advances in Math **167** (2002), 195-264.

[LLR00] F. Larsen, N. J. Laustsen, M. Rørdam, *An Introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts **49** (2000) CUP, Cambridge.

[Ped94] G.K. Pedersen, *Pullback and pushout constructions in C^*-algebra theory*, J. Funct. Anal. **167** (1999), 243–344.

[Pim95] M. V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Free probability theory, Fields Inst. Commun. **12** (1997), 189–212.

[Rief83] M. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), 301-333.

[Rief87] M. Rieffel, *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory **17** (1987), 237-254.

[Roh09] R. Rohde, K_1*-injectivity of* C^**-algebras*, Ph.D. thesis at Odense (2009)
[Ror97] M. Rørdam, *Stability of* C^**-algebras is not a stable property.*, Doc. Math. J. **2** (1997), 375–386.
[Ror03] M. Rørdam, *A simple* C^**-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), 109142.
[Ror04] M. Rørdam, *The stable and the real ranks of* $\mathcal{Z}$*-absorbing* C^**-algebras*, Internat. J. Math. **15** (2004), 1065–1084.
[TW07] A. Toms, W. Winter, *Strongly self-absorbing* C^**-algebras*, Trans. Amer. Math. Soc. **359** (2007), 3999-4029.
[Voi85] D. Voiculescu, *Symmetries of some reduced free product* C^**–algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory (Buşteni, 1983), Lecture Notes in Math. **112** (1985), 556–588.
[Win09] W. Winter, *Strongly self-absorbing* C^**-algebras are* $\mathcal{Z}$*-stable*, J. Noncommut. Geom. **5** (2011), 253-264.
[Wor95] S. L. Woronowicz, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam (1998), 845-884.

Etienne.Blanchard@math.jussieu.fr

IMJ, Bâtiment Sophie Germain, Case 7012, F–75013 Paris